

## S.T. YAU COLLEGE MATHEMATICS CONTESTS 2010

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ABSTRACT. In this paper, we give a reference answer to the Analysis and Differential Equations in S.T. Yau College Mathematics Contests 2010.

1. (a) Let  $\{x_k\}_{k=1}^n \subset (0, \pi)$ , and define

$$x = \frac{1}{n} \sum_{k=1}^n x_k.$$

Show that

$$\prod_{k=1}^n \frac{\sin x_k}{x_k} \leq \left( \frac{\sin x}{x} \right)^n.$$

*Proof.* Direct computations show

$$\left( \ln \frac{\sin x}{x} \right)'' = (\ln \sin x - \ln x)'' = \frac{-1}{\sin^2 x} + \frac{1}{x^2} > 0,$$

for all  $x \in (0, \pi)$ . Thus  $\ln \frac{\sin x}{x}$  is a convex function in  $(0, \pi)$ . Jensen's inequality then yields

$$\frac{1}{n} \sum_{k=1}^n \ln \frac{\sin x_k}{x_k} \leq \ln \frac{\sin x}{x}.$$

The exponential of this above inequality is the desired result.  $\square$

(b) From

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

calculate the integral  $\int_0^{\infty} \sin(x^2) dx$ .

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*Proof.* Consider the sector in  $\mathbf{R}^2$  enclosed by the following three curves

$$\begin{cases} I : & 0 \leq z \leq R, \\ II : & Re^{i\theta}, 0 \leq \theta \leq \frac{\pi}{4}, \\ III : & re^{i\frac{\pi}{4}}, 0 \leq r \leq R. \end{cases}$$

Cauchy's integration theorem then yields

$$0 = \left[ \int_I + \int_{II} + \int_{III} \right] e^{iz^2} dz. \quad (1)$$

Noticing

$$(i) \int_I e^{iz^2} dz = \int_0^R e^{ix^2} dx,$$

(ii)

$$\begin{aligned} \left| \int_{II} e^{iz^2} dz \right| &= \left| \int_0^{\frac{\pi}{4}} e^{iR^2 e^{2i\theta}} \cdot iRe^{i\theta} d\theta \right| \\ &\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta} d\theta \\ &\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \cdot \frac{2}{\pi} \cdot 2\theta} d\theta \\ &= \frac{\pi}{4R} (1 - e^{-R^2}) \\ &\rightarrow 0, \text{ as } R \rightarrow \infty, \end{aligned}$$

$$(iii) \int_{III} e^{iz^2} dz = - \int_0^R e^{ir^2 e^{i\frac{\pi}{2}}} \cdot e^{i\frac{\pi}{4}} dr = e^{i\frac{\pi}{4}} \int_0^R e^{-r^2} dr,$$

we have, by sending  $R \rightarrow \infty$  in (1), that

$$\int_0^{\infty} e^{ix^2} dx = e^{i\frac{\pi}{4}} \int_0^{\infty} e^{-r^2} dr.$$

Taking the imaginary part of this above equality gives

$$\int_0^{\infty} \sin(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

□

2. Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be any function. Prove that the set

$$C = \left\{ x_0 \in \mathbf{R}; f(x_0) = \lim_{x \rightarrow x_0} f(x) \right\}$$

is a  $G_\delta$ -set.

*Proof.* By definition,

$$C = \bigcap_{k=1}^{\infty} C_k,$$

where

$$C_k = \left\{ x_0 \in \mathbf{R}; \exists \delta_{x_0} > 0, \text{ s.t. } |x - x_0| < \delta_{x_0} \Rightarrow |f(x) - f(x_0)| < \frac{1}{k} \right\}$$

is an open set. In fact,

$$x_0 \in C_k \Rightarrow U(x_0, \delta_{x_0}) \subset C_k.$$

□

3. Consider the ODE

$$\dot{x} = -x + f(t, x),$$

where

$$\begin{cases} |f(t, x)| \leq \varphi(t) |x|, (t, x) \in \mathbf{R} \times \mathbf{R}, \\ \int_0^{\infty} \varphi(t) dt < \infty. \end{cases}$$

Prove that every solution approaches zero as  $t \rightarrow \infty$ .

*Proof.* For all  $t \in [0, \infty)$ , we have

$$\infty > \int_0^t \varphi(s) ds \geq \int_0^t \left| \frac{\dot{x}(s) + x(s)}{x(s)} \right| ds = \int_0^t \left| \frac{(e^s x(s))'}{e^s x(s)} \right| ds$$

$$\geq \left| \int_0^t d(e^s x(s)) \right| = |e^t x(t) - x(0)|.$$

Thus

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} e^{-t} \cdot [e^t x(t)] = 0.$$

□

4. Solve the PDE

$$\begin{cases} \Delta u = 0, & \text{in } \mathbf{R}^+ \times \mathbf{R}, \\ u = g, & \text{on } \{x_1 = 0\} \times \mathbf{R}, \end{cases}$$

where

$$g(x_2) = \begin{cases} 1, & \text{if } x_2 > 0, \\ -1, & \text{if } x_2 < 0. \end{cases}$$

*Proof.* It is standard (easy to verify) that

$$u(x) = \int_{\{y_1=0\} \times \mathbf{R}} u(y) \frac{\partial G}{\partial \mathbf{n}}(x, y) dS(y),$$

where

$$G(x, y) = \frac{1}{2\pi} [\ln|y - x| - \ln|y - \tilde{x}|]$$

is the Green's function for  $|x_1 > 0|$ , with  $\tilde{x}$  the reflection of  $x$  in the plane  $\{x_1 = 0\}$ .

Direct computations show

$$\begin{aligned} \frac{\partial G}{\partial \mathbf{n}}(x, y) &= -\frac{\partial G}{\partial y_1}(x, y) = -\frac{1}{2\pi} \left[ \frac{y_1 - x_1}{|y - x|^2} - \frac{y_1 + x_1}{|y - \tilde{x}|} \right] \\ &= -\frac{1}{2\pi} \frac{-2x_1}{|y - x_1|^2} ||y - x| = |y - \tilde{x}| \\ &= \frac{x_1}{\pi |y - x|^2}. \end{aligned}$$

Thus

$$\begin{aligned}
u(x) &= \int_{\{y_1=0\} \times \mathbf{R}} u(y) \frac{x_1}{\pi |y-x|^2} dS(y) \\
&= -\frac{x_1}{\pi} \int_{-\infty}^{\infty} \frac{g(y_2)}{x_1^2 + (y_2 - x_2)^2} dy_2 \\
&= -\frac{x_1}{\pi} \left[ \frac{1}{x_1} \int_{-\infty}^0 \frac{-1}{1 + \left| \frac{y_2 - x_2}{x_1} \right|^2} d \frac{y_2 - x_2}{x_1} + \frac{1}{x_1} \int_0^{\infty} \frac{1}{1 + \left( \frac{y_2 - x_2}{x_1} \right)^2} d \frac{y_2 - x_2}{x_1} \right] \\
&= -\frac{1}{\pi} \left[ -\arctan \frac{y_2 - x_2}{x_1} \Big|_{y_2=-\infty}^{y_2=0} + \arctan \frac{y_2 - x_2}{x_1} \Big|_{y_2=0}^{y_2=\infty} \right] \\
&= \frac{2}{\pi} \arctan \frac{x_2}{x_1}, \quad x = (x_1, x_2) \in \mathbf{R}^2.
\end{aligned}$$

□

5. Let  $K \in C([0, 1] \times [0, 1])$ . For  $f \in C[0, 1]$ , the space of continuous functions on  $[0, 1]$ , define

$$Tf(x) = \int_0^1 K(x, y) f(y) dy.$$

Prove that  $Tf \in C[0, 1]$ . Moreover,

$$\Omega = \left\{ Tf; \|f\|_{sup} \leq 1 \right\}$$

is precompact in  $C[0, 1]$ .

*Proof.* (a)  $Tf \in C[0, 1]$ .

$$\begin{aligned}
|Tf(x_1) - Tf(x_2)| &\leq \int_0^1 |K(x_1, y) - K(x_2, y)| |f(y)| dy \\
&\rightarrow 0, \text{ as } |x_1 - x_2| \rightarrow 0, \tag{2}
\end{aligned}$$

by the uniform continuity of  $K$  in  $x$  and  $y$ .

(b)  $\Omega$  is precompact in  $C[0, 1]$ .

This follows readily from

(i) the uniform boundedness of  $f \in \Omega$ :

$$\|f\|_{\text{sup}} \leq 1,$$

(ii) the equicontinuity of  $f \in \Omega$ , that is, (2),

(iii) and Ascoli-Azerá theorem.

□

6. Prove the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(x + 2n\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx},$$

for

$$f \in \mathcal{S}(\mathbf{R}) = \{f \in L^1_{\text{loc}}(\mathbf{R}); (1 + |x|^m) |f^{(n)}(x)| \leq C_{m,n}, \forall m, n \geq 0\}.$$

Here

$$\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-ix\xi} dx.$$

*Proof.* Define

$$h(x) = \sum_{n=-\infty}^{\infty} f(x + 2n\pi).$$

Then  $h$  is periodic with periodical  $2\pi$ . And hence the coefficients of its Fourier series are

$$\begin{aligned} a_k &= \frac{1}{2\pi} \int_0^{2\pi} h(x) e^{-ikx} dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(x + 2n\pi) e^{-ikx} dx \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2n\pi}^{2(n+1)\pi} f(y) e^{-ik(y-2n\pi)} dy \\ &= \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \hat{f}(k). \end{aligned}$$

Consequently,

$$\sum_{n=-\infty}^{\infty} f(x + 2n\pi) = h(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx} = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}.$$

□

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