# S.T. YAU COLLEGE MATHEMATICS CONTESTS 2010

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ABSTRACT. In this paper, we give a reference answer to the Analysis and Differential Equations in S.T. Yau College Mathematics Contests 2010.

1. (a) Let $\{x_k\}_{k=1}^n \subset (0, \pi)$, and define

$$x = \frac{1}{n} \sum_{k=1}^n x_i.$$

Show that

$$\prod_{k=1}^n \frac{\sin x_k}{x_k} \leq \left( \frac{\sin x}{x} \right)^n.$$

*Proof.* Direct computations show

$$\left( \ln \frac{\sin x}{x} \right)'' = (\ln \sin x - \ln x)'' = \frac{-1}{\sin^2 x} + \frac{1}{x^2} > 0,$$

for all $x \in (0, \pi)$. Thus $\ln \dfrac{\sin x}{x}$ is a convex function in $(0, \pi)$. Jensen's inequality then yields

$$\frac{1}{n} \sum_{k=1}^n \ln \frac{\sin x_k}{x_k} \leq \ln \frac{\sin x}{x}.$$

The exponential of this above inequality is the desired result. □

(b) From

$$\int_0^\infty e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\displaystyle\int_0^\infty \sin\left(x^2\right) \mathrm{d}x$.

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*Proof.* Consider the sector in $\mathbf{R}^2$ enclosed by the following three curves

$$\begin{cases} I: & 0 \le z \le R, \\ II: & Re^{i\theta},\ 0 \le \theta \le \frac{\pi}{4}, \\ III: & re^{i\frac{\pi}{4}},\ 0 \le r \le R. \end{cases}$$

Cauchy's integration theorem then yields

$$0 = \left[ \int_I + \int_{II} + \int_{III} \right] e^{iz^2}\,\mathrm{d}z. \tag{1}$$

Noticing

(i) $\displaystyle\int_I e^{iz^2}\,\mathrm{d}z = \int_0^R e^{ix^2}\,\mathrm{d}x,$

(ii)

$$\begin{aligned} \left| \int_{II} e^{iz^2}\mathrm{d}z \right| &= \left| \int_0^{\frac{\pi}{4}} e^{iR^2 e^{2i\theta}} \cdot iRe^{i\theta}\mathrm{d}\theta \right| \\ &\le R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta}\mathrm{d}\theta \\ &\le R \int_0^{\frac{\pi}{4}} e^{-R^2 \cdot \frac{2}{\pi} \cdot 2\theta}\mathrm{d}\theta \\ &= \frac{\pi}{4R}\left( 1 - e^{-R^2} \right) \\ &\to 0,\ \text{as } R \to \infty, \end{aligned}$$

(iii) $\displaystyle\int_{III} e^{iz^2}\,\mathrm{d}z = -\int_0^R e^{ir^2 e^{i\frac{\pi}{2}}} \cdot e^{i\frac{\pi}{4}}\mathrm{d}r = e^{i\frac{\pi}{4}} \int_0^R e^{-r^2}\mathrm{d}r,$

we have, by sending $R \to \infty$ in (1), that

$$\int_0^\infty e^{ix^2}\,dx = e^{i\frac{\pi}{4}} \int_0^\infty e^{-r^2}\,\mathrm{d}r.$$

Taking the imaginary part of this above equality gives

$$\int_0^\infty \sin(x^2)\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

□

2. Let $f : \mathbf{R} \to \mathbf{R}$ be any function. Prove that the set

$$C = \left\{ x_0 \in \mathbf{R};\ f(x_0) = \lim_{x \to x_0} f(x) \right\}$$

is a $G_\delta$-set.

*Proof.* By definition,

$$C = \cap_{k=1}^\infty C_k,$$

where

$$C_k = \left\{ x_0 \in \mathbf{R};\ \exists\, \delta_{x_0} > 0,\ s.t.\ |x - x_0| < \delta_{x_0} \Rightarrow |f(x) - f(x_0)| < \frac{1}{k} \right\}$$

is an open set. In fact,

$$x_0 \in C_k \Rightarrow U(x_0, \delta_{x_0}) \subset C_k.$$

□

3. Consider the *ODE*

$$\dot{x} = -x + f(t, x),$$

where

$$\begin{cases} |f(t, x)| \le \varphi(t)\,|x|,\ (t, x) \in \mathbf{R} \times \mathbf{R}, \\ \int^\infty \varphi(t)\mathrm{d}t < \infty. \end{cases}$$

Prove that every solution approaches zero as $t \to \infty$.

*Proof.* For all $t \in [0, \infty)$, we have

$$\infty\ >\ \int_0^t \varphi(s)\mathrm{d}s \ge \int_0^t \left| \frac{\dot{x}(s) + x(s)}{x(s)} \right| \mathrm{d}s = \int_0^t \frac{(e^s x(s))'}{e^s x(s)}\mathrm{d}s$$

$$\geq \left| \int_0^t \mathrm{d}\left(e^s x(s)\right) \right| = \left| e^t x(t) - x(0) \right|.$$

Thus

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} e^{-t} \cdot \left[ e^t x(t) \right] = 0.$$

$\square$

4. Solve the *PDE*

$$\begin{cases} \Delta u = 0, & \text{in } \mathbf{R}^+ \times \mathbf{R}, \\ u = g, & \text{on } \{x_1 = 0\} \times \mathbf{R}, \end{cases}$$

where

$$g(x_2) = \begin{cases} 1, & \text{if } x_2 > 0, \\ -1, & \text{if } x_2 < 0. \end{cases}$$

*Proof.* It is standard (easy to verfiy) that

$$u(x) = \int_{\{y_1=0\} \times \mathbf{R}} u(y) \frac{\partial G}{\partial \mathbf{n}}(x, y) \mathrm{d}S(y),$$

where

$$G(x, y) = \frac{1}{2\pi} \left[ \ln |y - x| - \ln |y - \tilde{x}| \right]$$

is the Green's function for $|x_1 > 0|$, with $\tilde{x}$ the reflection of $x$ in the plane $\{x_1 = 0\}$.

Direct computations show

$$\begin{aligned} \frac{\partial G}{\partial \mathbf{n}}(x, y) &= -\frac{\partial G}{\partial y_1}(x, y) = -\frac{1}{2\pi} \left[ \frac{y_1 - x_1}{|y - x|^2} - \frac{y_1 + x_1}{|y - \tilde{x}|} \right] \\ &= -\frac{1}{2\pi} \frac{-2x_1}{|y - x_1|^2} \ ||y - x| = |y - \tilde{x}|| \\ &= \frac{x_1}{\pi |y - x|^2}. \end{aligned}$$

Thus

$$
\begin{aligned}
u(x) &= \int_{\{y_1=0\}\times\mathbf{R}} u(y)\frac{x_1}{\pi\,|y-x|^2}\mathrm{d}S(y) \\
&= -\frac{x_1}{\pi}\int_{-\infty}^{\infty}\frac{g(y_2)}{x_1^2+(y_2-x_2)^2}\mathrm{d}y_2 \\
&= -\frac{x_1}{\pi}\left[\frac{1}{x_1}\int_{-\infty}^{0}\frac{-1}{1+\left|\frac{y_2-x_2}{x_1}\right|^2}\mathrm{d}\frac{y_2-x_2}{x_1}+\frac{1}{x_1}\int_{0}^{\infty}\frac{1}{1+\left(\frac{y_2-x_2}{x_1}\right)^2}\mathrm{d}\frac{y_2-x_2}{x_1}\right] \\
&= -\frac{1}{\pi}\left[-\arctan\frac{y_2-x_2}{x_1}\Big|_{y_2=-\infty}^{y_2=0}+\arctan\frac{y_2-x_2}{x_1}\Big|_{y_2=0}^{y_2=\infty}\right] \\
&= \frac{2}{\pi}\arctan\frac{x_2}{x_1},\ x=(x_1,x_2)\in\mathbf{R}^2.
\end{aligned}
$$

$\square$

5. Let $K \in C\left([0,1]\times[0,1]\right)$. For $f \in C[0,1]$, the space of continuous functions on $[0,1]$, define

$$
Tf(x) = \int_0^1 K(x,y)f(y)\mathrm{d}y.
$$

Prove that $Tf \in C[0,1]$. Moreover,

$$
\Omega = \left\{Tf;\ \|f\|_{sup} \le 1\right\}
$$

is precompact in $C[0,1]$.

*Proof.* (a) $Tf \in C[0,1]$.

$$
\begin{aligned}
|Tf(x_1)-Tf(x_2)| &\le \int_0^1 |K(x_1,y)-K(x_2,y)|\,|f(y)|\,\mathrm{d}y \\
&\to\ 0,\ \text{as } |x_1-x_2|\to 0, \tag{2}
\end{aligned}
$$

by the uniform continuity of $K$ in $x$ and $y$.

(b) $\Omega$ is precompact in $C[0,1]$.

This follows readily from

(i) the unform boundedness of $f \in \Omega$:

$$\|f\|_{\sup} \le 1,$$

(ii) the equicontinuity of $f \in \Omega$, that is, (2),

(iii) and Ascoli-Azerá theorem.

$\square$

6. Prove the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(x + 2n\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx},$$

for

$$f \in \mathcal{S}(\boldsymbol{R}) = \left\{ f \in L^1_{loc}(\boldsymbol{R}); \ (1 + |x|^m) \left| f^{(n)}(x) \right| \le C_{m,n}, \ \forall \, m, n \ge 0 \right\}.$$

Here

$$\hat{f}(\xi) = \int_{\boldsymbol{R}} f(x)e^{-ix\xi}\mathrm{d}x.$$

*Proof.* Define

$$h(x) = \sum_{n=-\infty}^{\infty} f(x + 2n\pi).$$

Then $h$ is periodic with periodical $2\pi$. And hence the coefficients of its Fourier series are

$$
\begin{aligned}
a_k &= \frac{1}{2\pi} \int_0^{2\pi} h(x)e^{-ikx}\mathrm{d}x = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(x + 2n\pi)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2n\pi}^{2(n+1)\pi} f(y)e^{-ik(y-2n\pi)}\mathrm{d}y \\
&= \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x = \hat{f}(k).
\end{aligned}
$$

Consequently,

$$\sum_{n=-\infty}^{\infty} f(x + 2n\pi) = h(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx} = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}.$$

$\square$

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