Charleton University Mathematics Journal
Volume **1**, Number **15**, December **2010**

Website: http://www.sciencenet.cn/u/zjzhang/
pp. **69-75**

S.T. YAU COLLEGE MATHEMATICS CONTESTS 2010

ZUJIN ZHANG

ABSTRACT. In this paper, we give a reference answer to the Analysis and Differential Equations in S.T. Yau College Mathematics Contests 2010.

1. (a) Let $\{x_k\}_{k=1}^n \subset (0,\pi)$, and define

$$x = \frac{1}{n}\sum_{k=1}^n x_i.$$

Show that

$$\prod_{k=1}^n \frac{\sin x_k}{x_k} \leq \left(\frac{\sin x}{x}\right)^n.$$

Proof. Direct computations show

$$\left(\ln\frac{\sin x}{x}\right)'' = (\ln\sin x - \ln x)'' = \frac{-1}{\sin^2 x} + \frac{1}{x^2} > 0,$$

for all $x \in (0,\pi)$. Thus $\ln\dfrac{\sin x}{x}$ is a convex function in $(0,\pi)$. Jensen's inequality then yields

$$\frac{1}{n}\sum_{k=1}^n \ln\frac{\sin x_k}{x_k} \leq \ln\frac{\sin x}{x}.$$

The exponential of this above inequality is the desired result. □

(b) From

$$\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\displaystyle\int_0^\infty \sin\left(x^2\right)\mathrm{d}x$.

Key words and phrases. S.T. Yau, College Mathematics Contests, analysis, differential equations.

Proof. Consider the sector in $\boldsymbol{R}^2$ enclosed by the following three curves

$$\begin{cases} I: & 0 \le z \le R, \\ II: & Re^{i\theta},\ 0 \le \theta \le \frac{\pi}{4}, \\ III: & re^{i\frac{\pi}{4}},\ 0 \le r \le R. \end{cases}$$

Cauchy's integration theorem then yields

$$0 = \left[\int_I + \int_{II} + \int_{III}\right] e^{iz^2}\mathrm{d}z. \tag{1}$$

Noticing

(i) $\displaystyle\int_I e^{iz^2}\mathrm{d}z = \int_0^R e^{ix^2}\mathrm{d}x$,

(ii)

$$\begin{aligned} \left|\int_{II} e^{iz^2}\mathrm{d}z\right| &= \left|\int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\theta}} \cdot iRe^{i\theta}\mathrm{d}\theta\right| \\ &\le R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\mathrm{d}\theta \\ &\le R\int_0^{\frac{\pi}{4}} e^{-R^2\cdot\frac{2}{\pi}\cdot 2\theta}\mathrm{d}\theta \\ &= \frac{\pi}{4R}\left(1-e^{-R^2}\right) \\ &\to 0,\ \text{as } R\to\infty, \end{aligned}$$

(iii) $\displaystyle\int_{III} e^{iz^2}\mathrm{d}z = -\int_0^R e^{ir^2e^{i\frac{\pi}{2}}} \cdot e^{i\frac{\pi}{4}}\mathrm{d}r = e^{i\frac{\pi}{4}}\int_0^R e^{-r^2}\mathrm{d}r$,

we have, by sending $R\to\infty$ in (1), that

$$\int_0^\infty e^{ix^2}dx = e^{i\frac{\pi}{4}}\int_0^\infty e^{-r^2}\mathrm{d}r.$$

Taking the imaginary part of this above equality gives

$$\int_0^\infty \sin(x^2)\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

□

2. Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be any function. Prove that the set

$$C = \left\{ x_0 \in \boldsymbol{R};\ f(x_0) = \lim_{x \to x_0} f(x) \right\}$$

is a G_δ-set.

Proof. By definition,

$$C = \cap_{k=1}^{\infty} C_k,$$

where

$$C_k = \left\{ x_0 \in \boldsymbol{R};\ \exists\, \delta_{x_0} > 0,\ s.t.\ |x - x_0| < \delta_{x_0} \Rightarrow |f(x) - f(x_0)| < \frac{1}{k} \right\}$$

is an open set. In fact,

$$x_0 \in C_k \Rightarrow U(x_0, \delta_{x_0}) \subset C_k.$$

□

3. Consider the *ODE*

$$\dot{x} = -x + f(t, x),$$

where

$$\begin{cases} |f(t,x)| \le \varphi(t)\,|x|\,,\ (t,x) \in \boldsymbol{R} \times \boldsymbol{R}, \\ \int^{\infty} \varphi(t)\mathrm{d}t < \infty. \end{cases}$$

Prove that every solution approaches zero as $t \to \infty$.

Proof. For all $t \in [0, \infty)$, we have

$$\infty \quad > \quad \int_0^t \varphi(s)\mathrm{d}s \ge \int_0^t \left| \frac{\dot{x}(s) + x(s)}{x(s)} \right| \mathrm{d}s = \int_0^t \left| \frac{(e^s x(s))'}{e^s x(s)} \right| \mathrm{d}s$$

$$\geq \left| \int_0^t \mathrm{d}\left(e^s x(s)\right) \right| = \left| e^t x(t) - x(0) \right|.$$

Thus

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{-t} \cdot \left[e^t x(t)\right] = 0.$$

□

4. Solve the *PDE*

$$\begin{cases} \Delta u = 0, & \text{in } \boldsymbol{R}^+ \times \boldsymbol{R}, \\ u = g, & \text{on } \{x_1 = 0\} \times \boldsymbol{R}, \end{cases}$$

where

$$g(x_2) = \begin{cases} 1, & \text{if } x_2 > 0, \\ -1, & \text{if } x_2 < 0. \end{cases}$$

Proof. It is standard (easy to verfiy) that

$$u(x) = \int_{\{y_1=0\}\times\boldsymbol{R}} u(y) \frac{\partial G}{\partial \boldsymbol{n}}(x,y)\mathrm{d}S(y),$$

where

$$G(x,y) = \frac{1}{2\pi}\left[\ln|y-x| - \ln|y-\tilde{x}|\right]$$

is the Green's function for $\{x_1 > 0\}$, with $\tilde{x}$ the reflection of x in the plane $\{x_1 = 0\}$.

Direct computations show

$$\begin{aligned} \frac{\partial G}{\partial \boldsymbol{n}}(x,y) &= -\frac{\partial G}{\partial y_1}(x,y) = -\frac{1}{2\pi}\left[\frac{y_1 - x_1}{|y-x|^2} - \frac{y_1 + x_1}{|y-\tilde{x}|}\right] \\ &= -\frac{1}{2\pi}\frac{-2x_1}{|y-x_1|^2} \quad (|y-x| = |y-\tilde{x}|) \\ &= \frac{x_1}{\pi|y-x|^2}. \end{aligned}$$

Thus

$$\begin{aligned}
u(x) &= \int\limits_{\{y_1=0\}\times\boldsymbol{R}} u(y)\frac{x_1}{\pi\,|y-x|^2}\mathrm{d}S(y)\\
&= -\frac{x_1}{\pi}\int\limits_{-\infty}^{\infty}\frac{g(y_2)}{x_1^2+(y_2-x_2)^2}\mathrm{d}y_2\\
&= -\frac{x_1}{\pi}\left[\frac{1}{x_1}\int\limits_{-\infty}^{0}\frac{-1}{1+\left|\frac{y_2-x_2}{x_1}\right|^2}\mathrm{d}\frac{y_2-x_2}{x_1}+\frac{1}{x_1}\int\limits_{0}^{\infty}\frac{1}{1+\left(\frac{y_2-x_2}{x_1}\right)^2}\mathrm{d}\frac{y_2-x_2}{x_1}\right]\\
&= -\frac{1}{\pi}\left[-\arctan\frac{y_2-x_2}{x_1}\Bigg|_{y_2=-\infty}^{y_2=0}+\arctan\frac{y_2-x_2}{x_1}\Bigg|_{y_2=0}^{y_2=\infty}\right]\\
&= \frac{2}{\pi}\arctan\frac{x_2}{x_1},\ x=(x_1,x_2)\in\boldsymbol{R}^2.
\end{aligned}$$

□

5. Let $K\in C([0,1]\times[0,1])$. For $f\in C[0,1]$, the space of continuous functions on $[0,1]$, define

$$Tf(x)=\int\limits_0^1 K(x,y)f(y)\mathrm{d}y.$$

Prove that $Tf\in C[0,1]$. Moreover,

$$\Omega=\left\{Tf;\ \|f\|_{sup}\le 1\right\}$$

is precompact in $C[0,1]$.

Proof. (a) $Tf\in C[0,1]$.

$$\begin{aligned}
|Tf(x_1)-Tf(x_2)| &\le \int\limits_0^1 |K(x_1,y)-K(x_2,y)|\,|f(y)|\,\mathrm{d}y\\
&\to 0,\ \text{as}\ |x_1-x_2|\to 0,
\end{aligned}\tag{2}$$

by the uniform continuity of K in x and y.

(b) Ω is precompact in $C[0,1]$.

This follows readily from

(i) the unform boundedness of $f \in \Omega$:

$$\|f\|_{\sup} \leq 1,$$

(ii) the equicontinuity of $f \in \Omega$, that is, (2),

(iii) and the Ascoli-Azerá theorem.

□

6. Prove the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(x+2n\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx},$$

for

$$f \in \mathcal{S}(\boldsymbol{R}) = \left\{ f \in L^1_{loc}(\boldsymbol{R});\ (1+|x|^m) \left| f^{(n)}(x) \right| \leq C_{m,n},\ \forall\, m,n \geq 0 \right\}.$$

Here

$$\hat{f}(\xi) = \int_{\boldsymbol{R}} f(x) e^{-ix\xi} \mathrm{d}x.$$

Proof. Define

$$h(x) = \sum_{n=-\infty}^{\infty} f(x+2n\pi).$$

Then h is periodic with periodical 2π. And hence the coefficients of its Fourier series are

$$\begin{aligned} a_k &= \frac{1}{2\pi} \int_0^{2\pi} h(x) e^{-ikx} \mathrm{d}x = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(x+2n\pi) e^{-ikx} \mathrm{d}x \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2n\pi}^{2(n+1)\pi} f(y) e^{-ik(y-2n\pi)} \mathrm{d}y \\ &= \int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x = \hat{f}(k). \end{aligned}$$

Consequently,

$$\sum_{n=-\infty}^{\infty} f(x+2n\pi) = h(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx} = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}.$$

□

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com