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# Influences of Gate Operation Errors in the Quantum Counting Algorithm

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**Abstract** In this article, the error analysis in the quantum counting algorithm is investigated. It has been found that the random error plays as important a role as the systematic error does in the phase inversion operations. Both systematic and random errors are important in the Hadamard transformation. This is quite different from the Grover algorithm and the Shor algorithm.

**Keywords** quantum counting, Grover algorithm, gate error analysis

## 1 Introduction

Shor devised a quantum computing algorithm to speed up the factorization problem exponentially[1]. Grover algorithm was put forward to search a marked item out of an unsorted database. It is one milestone in quantum computation that speeds up this sort of problem quadratically.

In the original Grover algorithm[2], it focuses on searching a single item from an unsorted database. It is certain that $O(\sqrt{N})$ steps should be carried out[2]. However, usually there may be more than one marked items in the database. It has been shown that $O(\sqrt{\frac{N}{M}})$ steps are needed to find one of the marked items, where $M$ is the number of marked items[3]. As far as the algorithm itself is concerned, the number of steps should be known in advance, otherwise the algorithm will not display its superiority to its classical counterpart. This is solved by using the quantum counting algorithm[4] beforehand, which has recently been implemented in nuclear magnetic resonance system[5].

There are errors in the gate operations. These errors will affect the performance of an algorithm. Error analysis for Grover algorithm and Shor algorithm have been carried out[6−10]. It has been found that in Grover algorithm, systematic phase mismatching[11−14] and random errors in the Hadamard gate are dominant. Those errors set a stringent restriction on the size of the database[6]. In contrast, random error in the Shor algorithm has a more serious influence than systematic errors[7].

It is helpful to analyze the effect of errors in the counting algorithm. This is important for its practical implementation. It also provides us with hints that to which error we should give more consideration. In this article, we investigate the influence of errors in the counting algorithm.

## 2 Analysis of Phase Mismatching Errors

The quantum network for the counting algorithm is shown in Fig.1. It has two parts, the first stage and inverse quantum Fourier transformation. The first stage matches the eigenvalues of the searching operation $\boldsymbol{Q}$ and its powers to the $t$ workspace qubits on the top where the information about the number of marked items is mapped.

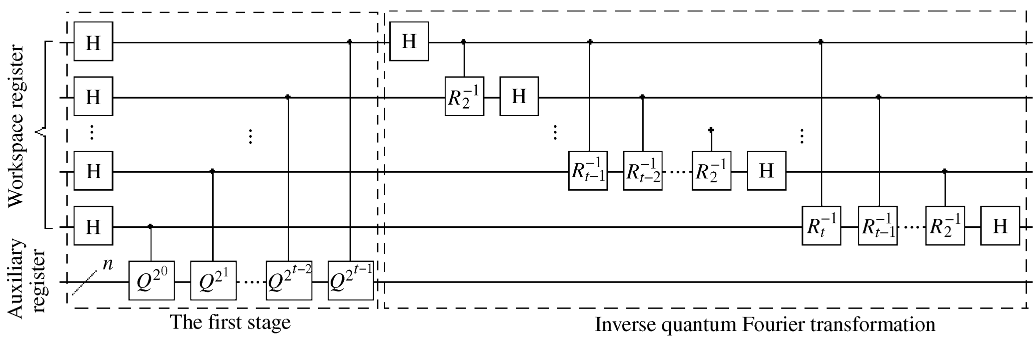


Fig.1. Quantum counting circuit. (It contains the first stage and inverse quantum Fourier transformation. The circuit consists of $t+n$ qubits, $t$-qubit on the top as workspace register, and $n$-qubit at the bottom as auxiliary register.)

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The counting algorithm uses the searching operator in the Grover algorithm, which includes four steps[2]: 1) a phase inversion of the marked state $|\tau\rangle$,

$$I_\tau = I - 2\sum_{k=1}^{M} |\tau_k\rangle\langle\tau_k|, \tag{1}$$

where the summation is over all possible marked states; 2) a Hadamard transformation $W$; 3) a phase inversion of all the states except $|0\rangle$,

$$-I_0 = -(I - 2|0\rangle\langle 0|); \tag{2}$$

4) again Hadamard transformation $W$. Thus, the iteration can be written as

$$\boldsymbol{Q} = -WI_0WI_\tau. \tag{3}$$

The standard Grover search operator can be viewed as a rotation in a two-dimension space spanned by the following two orthonormal bases

$$\begin{cases} |1\rangle = \frac{1}{\sqrt{N-M}}\sum_{i\neq\tau}|i\rangle, \\ |2\rangle = \frac{1}{\sqrt{M}}\sum_{k=1}^{M}|\tau_k\rangle. \end{cases} \tag{4}$$

As a consequence, the operator $\boldsymbol{Q}$ can be written as

$$\boldsymbol{Q} = \begin{pmatrix} \cos 2\beta & -\sin 2\beta \\ \sin 2\beta & \cos 2\beta \end{pmatrix}, \tag{5}$$

while the initial state is represented as $|\psi_i\rangle = \cos\beta|1\rangle + \sin\beta|2\rangle$, where $\tan\beta = \sqrt{\frac{M}{N-M}}$. It is obvious that the matrix $\boldsymbol{Q}$ has two eigen-values $e^{i2\beta}$ and $e^{-i2\beta}$ corresponding to two eigen states

$$\begin{cases} |a\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle), \\ |b\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle), \end{cases} \tag{6}$$

respectively.

To study in the phase inversion, the inversion operations are written in the following form

$$\begin{cases} I_\tau = I - (1-e^{i\theta})\sum_{k=1}^{M}|\tau_k\rangle\langle\tau_k|, \\ I_0 = I - (1-e^{i\varphi})|0\rangle\langle 0|, \end{cases} \tag{7}$$

where $\theta = \varphi = \pi$ in the perfect gate operations. Consequentially, we make a unitary transformation from $|1\rangle$ and $|2\rangle$ bases to $|a\rangle$ and $|b\rangle$ bases. Then, the operator $\boldsymbol{Q}$ has the form

$$\boldsymbol{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \tag{8}$$

where $Q_{11} = -e^{i\theta} + (e^{i\theta}-1)\sin^2\beta + i(e^{i\theta}-1)\sin\beta\cos\beta$, $Q_{12} = -e^{i\varphi}(e^{i\theta}-1)\sin\beta\cos\beta + ie^{i\varphi}[1+(e^{i\theta}-1)\sin^2\beta]$, $Q_{21} = -e^{i\theta} + (e^{i\theta}-1)\sin^2\beta - i(e^{i\theta}-1)\sin\beta\cos\beta$, $Q_{22} = -e^{i\varphi}(e^{i\theta}-1)\sin\beta\cos\beta - ie^{i\varphi}[1+(e^{i\theta}-1)\sin^2\beta]$.

In practice, in a database the number of solutions $M$ is much less than total number $N = 2^n$. As a good approximation, $\sin\beta \approx \beta$ and $\cos\beta \approx 1$. After dropping off an overall phase $-e^{\frac{i}{2}(\theta+\varphi)}$, we make an approximation that $\theta \approx \pi$ and $\varphi \approx \pi$. Thus,

$$\begin{aligned} \boldsymbol{Q} &\approx \frac{1}{2}\begin{pmatrix} \cos\delta - i2\beta & i\sin\delta \\ i\sin\delta & \cos\delta + i2\beta \end{pmatrix} \\ &= \cos\delta\cdot I - i2\beta\cdot\sigma_z + i\sin\delta\cdot\sigma_x \\ &\approx I - i2\beta\cdot\sigma_z + \delta\cdot\sigma_x \end{aligned} \tag{9}$$

where $\delta = \frac{1}{2}(\theta - \varphi)$ stands for the phase mismatching in the Grover iteration and $\sigma_x$, $\sigma_z$ are Pauli operators. Moreover, $\boldsymbol{Q}$ can be simplified as $\boldsymbol{Q} \approx \boldsymbol{I} + i\boldsymbol{G} \approx e^{i\boldsymbol{G}}$, where $\boldsymbol{G} = -2\beta\cdot\sigma_z + \delta\cdot\sigma_x$. Therefore,

$$\begin{aligned} \boldsymbol{Q}^j &= e^{ij\boldsymbol{G}} = \boldsymbol{I} + \sum_{n=1}^{\infty}\frac{1}{n!}(ij\boldsymbol{G})^n \\ &= \begin{pmatrix} \cos j\lambda - \frac{i}{\lambda}2\beta\sin j\lambda & \frac{i}{\lambda}\delta\sin j\lambda \\ \frac{i}{\lambda}\delta\sin j\lambda & \cos j\lambda + \frac{i}{\lambda}2\beta\sin j\lambda \end{pmatrix}, \end{aligned} \tag{10}$$

where $\lambda = \sqrt{\delta^2 + 4\beta^2}$.

As shown in Fig.1 in [3], the initial state of the $t$-qubit workspace register is $\bigotimes_{i=1}^{t}|0\rangle$ and $|\psi_i\rangle$ for the auxiliary register. The final state after the inverse quantum Fourier transformation is

$$\begin{aligned} |\psi_f\rangle &= \frac{1}{\sqrt{2}}e^{i\beta}|a\rangle\bigotimes_{j=1}^{t}\frac{1}{\sqrt{2}}[|0\rangle + T_{j-1}|1\rangle] \\ &\quad + \frac{1}{\sqrt{2}}e^{-i\beta}|b\rangle\bigotimes_{j=1}^{t}\frac{1}{\sqrt{2}}[|0\rangle + S_{j-1}|1\rangle] \\ &= \frac{1}{\sqrt{2}}e^{i\beta}|a\rangle\bigotimes\sum_{k=0}^{2^t-1}U_k|k\rangle \\ &\quad + \frac{1}{\sqrt{2}}e^{-i\beta}|b\rangle\bigotimes\sum_{k=0}^{2^t-1}V_k|k\rangle, \end{aligned} \tag{11}$$

where

$$\begin{cases} T_0 = \cos\lambda + \frac{i}{\lambda}(\delta e^{-i2\beta} + 2\beta)\sin\lambda, \\ T_j = \cos 2^j\lambda + \frac{i}{\lambda}(\delta + 2\beta)\sin 2^j\lambda, \\ S_0 = \cos\lambda + \frac{i}{\lambda}(\delta e^{i2\beta} - 2\beta)\sin\lambda, \\ S_j = \cos 2^j\lambda + \frac{i}{\lambda}(\delta - 2\beta)\sin 2^j\lambda, \end{cases} \tag{12}$$

for $j = 1, 2, \ldots, t-1$. In the second line of (11),

$$\begin{cases} U_k = \frac{1}{2^t}\prod_{j=1}^{t}[1 + (-1)^{k_j}T_{j-1}e^{2\pi i\sum_{l=j+1}^{t}k_l/2^{l-j+1}}], \\ V_k = \frac{1}{2^t}\prod_{j=1}^{t}[1 + (-1)^{k_j}S_{j-1}e^{2\pi i\sum_{l=j+1}^{t}k_l/2^{l-j+1}}], \end{cases} \tag{13}$$

where $k = k_1k_2\ldots k_t$ is the binary form of $k$. Afterwards, measurement is taken in the auxiliary register.

It is clear that its state will collapse into either $|a\rangle$ or $|b\rangle$. On condition that $|a\rangle$ is obtained, in the workspace register, the probability of obtaining authentic $M$ is $P = |U_{k_M}|^2 / \sum_{k=0}^{2^t-1} |U_k|^2$, where $k_M = \lceil \frac{2\beta}{2\pi} 2^t \rceil$ and $\lceil x \rceil$ refers to the rounding to the nearest integer. Two different error models are considered. One is the systematic error in which the phase mismatching $\delta$ is considered as a constant, while the other is the random error where $\delta$ is treated as an random variable with a Gaussian distribution. We assume they all have mean 0 but different standard deviations $s$ in order to see its effect.

In Fig.2, the cases with $M = 0$ and $M = 5$ are studied with different numbers of $n = \log_2 N$, where $N = 2^n$ is the number of items in the database. It is seen that the success probability falls exponentially along with the argument $n$. When $M = 0$, the probability falls monotonously, while in contrast there are fluctuations in Fig.2(b). Furthermore, the probability for $\delta = 0$ in Fig.2(a) remains steadily at unity in contrast to its counterpart in Fig.2(b). This sharp contrast between $M = 0$ and $M = 5$ is due to the finite number of qubits for obtaining the irrational number $\frac{2\beta}{2\pi} 2^t$. In the case of $M = 0$, no matter how the accuracy $t$ varies, $k_M = \lceil \frac{2\beta}{2\pi} 2^t \rceil$ and $\lceil \frac{2\beta}{2\pi} 2^t \rceil$ are all equal to zero. However, if $M$ is greater than zero, $k_M = \lceil \frac{2\beta}{2\pi} 2^t \rceil$ is always different from $\lceil \frac{2\beta}{2\pi} 2^t \rceil$, because of the non-zero value of $\beta$.

When the random error is taken into account, the behavior of the success probability is similar. We have plotted the cases with $M = 0$ and 5 in Fig.3 respectively. The behavior of the mean values of $P$ is analogous to the one with systematic errors.

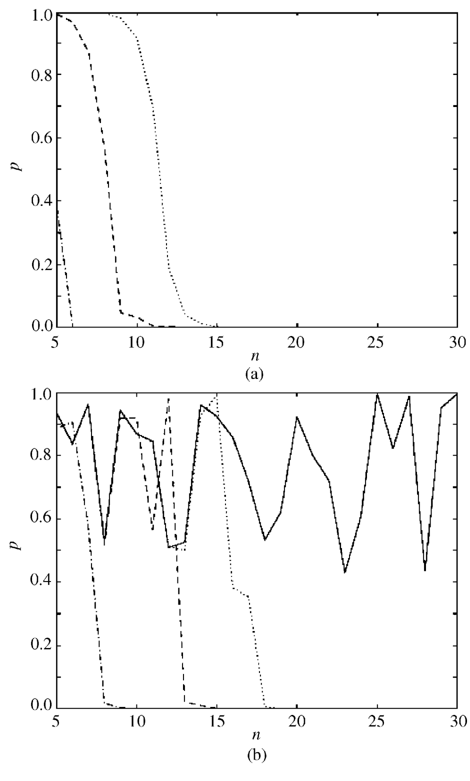

Fig.2. Systematic error in phase inversion: Estimate (a) $M = 0$ (b) $M = 5$ with $t = n$ qubits. (Dashed and dotted line for $\delta = 0.1$, dashed line for $\delta = 0.01$, dotted line for $\delta = 0.001$, bold solid line for $\delta = 0$.)

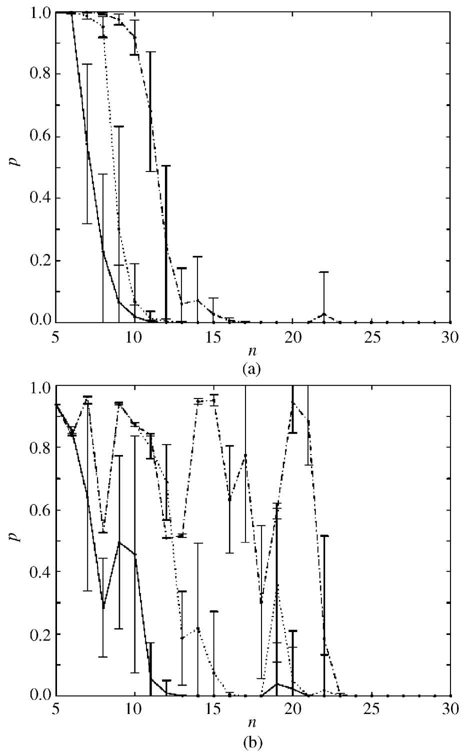

Fig.3. Random error in phase inversion: Estimate (a) $M = 0$ (b) $M = 5$ with $t = n$ qubits. (where $\delta$ is stochastically distributed around 0 with standard deviation $s$, solid line for $s = 0.1$, dotted line for $s = 0.01$, dashed and dotted line for $s = 0.001$.)

In Figs. 2 and 3, the success probability decreases exponentially with respect to the increase in $n$. With comparison to the case of Grover algorithm[6], there is a distinct difference. In [6], it was discovered that the systematic error plays a more important role than the random error. That is due to the sinusoidal and cosinusoidal terms of $T_j$ in (13). This can be understood in the following way: in the quantum counting algorithm, the searching operator is executed exponentially in the first stage in Fig.1, therefore the effect of random errors in the searching operator $\boldsymbol{Q}$ accumulates and makes the random errors as important as the systematic errors.

## 3 Analysis in Hadamard Transformation

To see the effect of errors in the Hadamard gate, we assume perfect phase inversion operations, and then (8) can be written as

$$\boldsymbol{Q} = \begin{pmatrix} e^{-i2\beta} & 0 \\ 0 & e^{i2\beta} \end{pmatrix}. \tag{14}$$

Due to the effects of gate errors in Hadamard transformation, the Grover iteration is not within a two-dimensional space any more. In [6], it was pointed out that the space after an iteration would change to another space spanned by $|1'\rangle = |1\rangle$ and $|2'\rangle = (1-\delta_1)|2\rangle$, where $|1\rangle$ and $|2\rangle$ are the bases before the iteration. In this view, if a Grover iteration is applied to an arbitrary state $|\psi\rangle = A_1|1\rangle+A_2|2\rangle = \frac{1}{\sqrt{2}}(A_1+iA_2)|a\rangle+\frac{1}{\sqrt{2}}(A_1-iA_2)|b\rangle$, then

$$\boldsymbol{Q}|\psi\rangle = \begin{pmatrix} e^{-i2\beta} & 0 \\ 0 & e^{i2\beta} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} A_1 - iA_2(1-\delta_1) \\ A_1 + iA_2(1-\delta_1) \end{pmatrix}. \tag{15}$$

Thus, in the space spanned by $|1\rangle$ and $|2\rangle$, $\boldsymbol{Q}$ can be written as

$$\boldsymbol{Q} = \begin{pmatrix} e^{-i2\beta}\Big(1-\dfrac{\delta_1}{2}\Big) & e^{-i2\beta}\dfrac{\delta_1}{2} \\ e^{i2\beta}\dfrac{\delta_1}{2} & e^{i2\beta}\Big(1-\dfrac{\delta_1}{2}\Big) \end{pmatrix}. \tag{16}$$

In Fig.4, the systematic errors are studied. It is noted that the systematic errors in the case of $M = 0$ affect the algorithm very slightly. In Fig.4(b), where $M = 5$, we see that even without any errors, the success rate of the counting algorithm is a fluctuating function of the number of items in the database. As more random errors come in, the success rate is reduced greatly. The larger the error is, the bigger and earlier the reduction in the success probability appears.

The random errors are also studied and are shown in Fig.5. As expected, there are fluctuations in the success probability due to the randomness of the errors. Taking the average over 500 calculations, we see that the behavior of the mean values are much the same as those for the systematic errors. The success probability decreases exponentially as $n$ grows large with the apparent exception of $M = 0$, which stays close to unity in Fig.5(a), and Fig.4(a) as well.

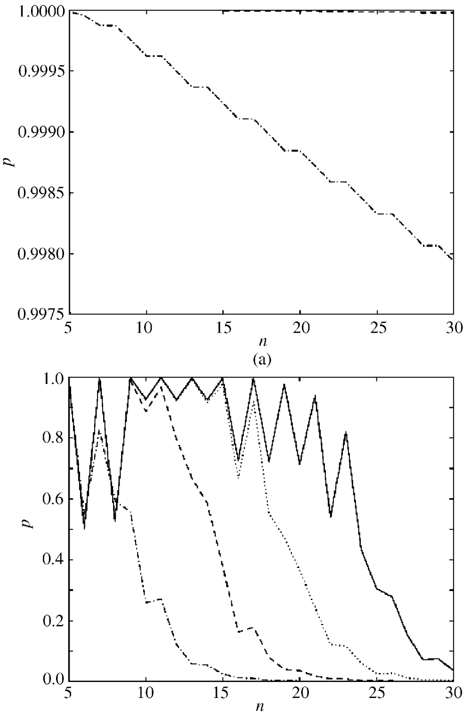

Fig.4. Systematic error in Hadamard transformation: Estimate (a) $M = 0$ (b) $M = 5$ with $t = \frac{2}{3}n$ qubits. (Dashed and dotted line for $\delta = 0.1$, dashed line for $\delta = 0.01$, dotted line for $\delta = 0.001$, solid line for $\delta = 0$.)

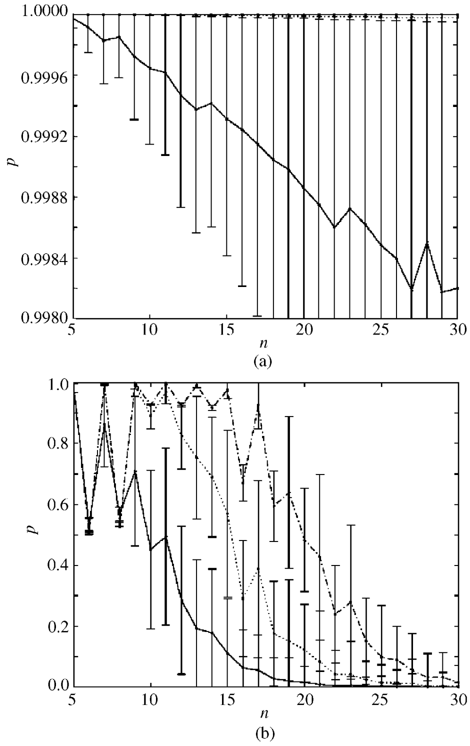

Fig.5. Random error in Hadamard transformation: Estimate (a) $M = 0$ (b) $M = 5$ with $t = \frac{2}{3}n$ qubits. ($\delta's$ average values are all set as 0 with three different standard deviations, solid line for $s = 0.1$, dotted line for $s = 0.01$, dashed and dotted line for $s = 0$.)

## 4 Summary

In this paper, we have studied the effects of gate errors in the quantum counting algorithm. Two error models, the random error and the systematic error, have been considered. Compared with the Grover algorithm, both systematic error and random error play important parts in affecting the performance of the counting algorithm. This is not surprising taking the fact that the counting algorithm combines Grover algorithm and the phase estimation algorithm. The phase estimation algorithm is very close to the Shor algorithm where the random error in the gate operation plays an important role. On the contrary, the systematic error in Grover algorithm plays a dominant role. The counting algorithm inherits from both algorithms and hence both random and systematic errors cannot outstrip the other side.

The result of this study suggests that one has to make a very delicate balance between the accuracy requirement in estimating the number of marked items and the errors tolerable in the counting algorithm. On the one hand, it is good to have a high accuracy in estimating the number of marked items in the database. On the other hand, as the accuracy increases, the errors in the algorithm also increases rapidly. This has been demonstrated in numerical simulations which has not shown here.

Judging from the analysis made above, we can conclude that in the case of gate imperfections, either the random error or the systematic error, the success probability for the quantum counting algorithm falls exponentially along with the increase of the number of qubits $n$ or the accuracy $t$. Errors in the quantum counting algorithm should be eliminated as much as possible in order to run it successfully. In other words, quantum error correction is more badly needed than the Grover algorithm or the Shor algorithm.

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## References

[1] Shor P W. Algorithms for quantum computation: Discrete logarighms and factoring. In *Proc. Symposium on the Foundings of Computer Science*, Los Alamitos, California, 1994, pp.124–134.

[2] Grover L K. Quantum mechanics helps in searching for a needle in a haystack. *Phys. Rev. Lett.*, 1997, 79(2): 325–328.

[3] Nielsen M A, Chuang I L. Quantum Computation and Quantum Information. Cambridge: Cambridge University Press, 2000.

[4] Boyer M, Brassard G, Høyer P, Tapp A. Tight bounds on quantum searching. *Fortsch. Phys.,* 1998, 46(4-5): 493–505.

[5] Lee Jae-Seung, Kim Jaehyun, Cheong Yongwook, Lee Soonchil. Implementation of phase estimation and quantum counting algorithms on an NMR quantum-information processor. *Phys. Rev. A*, 2002, 66(4): 042316.

[6] Long G L, Li Y S, Zhang W L, Tu C C. Dominant gate imperfection in Grover's quantum search algorithm. *Phys. Rev. A*, 2000, 61(4): 042305.

[7] Guo H, Long G L, Sun Y. Effects of imperfect gate operations in Shor's prime factorization algorithm. *J. Chin. Chem. Soc.*, 2001, 48: 449–454.

[8] Shenvi Neil, Brown K R, Whaley K B. Effects of a random noisy oracle on search algorithm complexity. *Phys. Rev. A*, 2003, 68(5): 052313.

[9] Niwa Jumpei, Matsumoto Keiji, Imai Hiroshi. General-purpose parallel simulator for quantum computing. *Phys. Rev. A*, 2002, 66(6): 062317.

[10] Wei L F, Li X, Hu X D, Noril Franco. Effects of dynamical phases in Shor's factoring algorithm with operational delays. *Phys. Rev. A*, 2005, 71(2): 022317.

[11] Long G L, Zhang W L, Li Y S, Niu L. Arbitrary phase rotation of the marked state cannot be used for Grover's quantum search algorithm. *Commun. Theor. Phys.*, 1999, 32: 335–338.

[12] Long G L, Li Y S, Zhang W L, Niu L. Phase matching in quantum searching. *Phys. Lett. A*, 1999, 262(1): 27–34.

[13] Long G L, Li X, Sun Y. Phase matching condition for quantum search with a generalized initial state. *Phys. Lett. A*, 2002, 294(3-4): 143–152.

[14] Long G L. Grover algorithm with zero theoretical failure rate. *Phys. Rev. A*, 2001, 64(2): 022307.