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The relation between properties of Gentile statistics and fractional statistics of anyon

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ABSTRACT

In this paper, we discuss the relationship of two kinds of intermediate-statistics, the Gentile statistics and the fractional statistics of anyons. The anyon winding number representation is introduced. We construct the transformation between anyon winding number representation and the occupation number representation of particles of Gentile statistics. We study intermediate-statistics quantum bracket and coherent states for anyons in the winding number representation. We demonstrate that anyons can be simulated by Gentile statistics with a geometric phase.

1. Introduction

It is well-known that there are two kinds of particles in nature, bosons and fermions. Bosons obey Bose–Einstein statistics, and the wave function of many bosons is symmetric and the maximum occupation number in a given state is infinity. Fermions obey Fermi–Dirac statistics, the wave function of many fermions is antisymmetric and the maximum occupation number in a state is one.

Besides these two kinds of statistics there is a third type of statistics, namely, intermediate statistics. There are two kinds of intermediate statistics, the Gentile statistics [1–4] and the fractional statistics of anyons [5–7]. Gentile statistics is a kind of intermediate statistics where the maximum occupation number of particles in one state is neither infinity nor one, but a finite integer n. Thus the boson and fermion statistics can be considered as two special cases of Gentile statistics. When n equals infinity, the relation of operators becomes commutative and Gentile statistics returns to Bose–Einstein statistics. When n equals one, the relation of operators becomes anticommutative and Gentile statistics returns to Fermi–Dirac statistics. Studies show that there exist real physical systems which obey Gentile statistics, such as magnons, spin waves excited from the Heisenberg magnetic system [8], and the result compares well with the experimental data for EuO.

Wilczek pointed out another group of intermediate statistics, — the statistics of anyons [5–7]. For many anyons, the wave function will change a phase $2\pi m+2\pi k\alpha$ when two identical anyons are braided [9,10], where α is the statistical parameter, k is the winding number of one particle braiding around another and m is an integer. Since the statistical parameter is fractional, the intermediate statistics of anyons is also called fractional statistics [11–13]. It is related to many interesting phenomena, e.g., Fermi gas superfluid [14–16], the fractional quantum Hall effect and high-T_c superconductivity [17]. Kitaev [18] gave an exactly solvable model of anyons and constructed the theories for both Abelian and non-Abelian anyons. Anyonic excitations in a two-dimensional system can be considered as a model of quantum computer [19].

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The particles of Gentile statistics and anyons are both intermediate statistics particles, they present two different characteristics of the intermediate statistics. One features the intermediate statistics using the maximum occupation number, while the other employs the symmetry property of wave functions under braiding. What we are usually familiar with is the description of the occupation number representation (ONR) of particles of Gentile statistics, for instance general results for the operation laws of operators, coherent states of particles of Gentile statistics, the Hamilton and energy spectrum of oscillator and so on were derived [20] using the intermediate statistics quantum bracket (ISQB) [21], an improved operational form for the maximum occupation number of Gentile statistics.

In recent years, more and more studies on intermediate statistics have been performed [22–26], and clear evidences of intermediate statistics have been found. The properties of each intermediate statistics have been studied. It is an interesting subject to study the relation between these two intermediate statistics. Here in this paper we treat the anyons as superpositions of states of particles of Gentile statistics. Anyons are characterized by the winding number representation (WNR). By constructing the transformation between WNR and ONR, the properties of anyons can be understood in a different point of view. The discussion may give another perspective to further study of intermediate statistics systems.

This paper is organized as follows. In Section 2 we give a brief introduction of WNR. In Section 3, we propose a realization of anyons using particles of Gentile statistics in the WNR. Then in Section 4 we present a method to simulate anyons by the geometric phase. At the end of this paper, we give a brief summary in Section 5.

2. The winding number representation

As we know, the wave function will change a phase factor $\exp(\mathrm{i}2\pi k\alpha)$ when two identical anyons are braided [9,10]. Here we ignore the part of integer without loss of generality. The phase factor exhibits the topology property of anyons. α is a kind of inherent property of anyons, and different anyons have different statistics parameters. In WNR, the basic vector is labeled by $|k\rangle_\alpha$. Here we propose that the WNR state $|k\rangle_\alpha$ is related to the basic vector of the ONR of Gentile statistics, where n is the maximum occupation number and ν is the occupation number of a quantum state, $|\nu\rangle_n$ with some restriction.

First we look the two extreme cases in Gentile statistics. When $n \to \infty$ and $n \to 1$, Gentile statistics returns to Bose–Einstein statistics and Fermi–Dirac statistics respectively. Correspondingly, for anyons, when the statistics parameter $\alpha = 0$ and $\alpha = 1/2$, anyons fractional statistics return to bosons and fermions respectively. For the generalized exclusion principle for fractional statistics, they satisfy $n \leq 1/\alpha$ [27]. Considering these, we propose to relate anyons and the particles in the Gentile statistics via

$$n = \frac{1}{\alpha} - 1, \tag{1}$$

and the intermediate bracket phase and the braiding phase the same

$$\mathrm{e}^{\frac{\mathrm{i}2\pi\nu}{n+1}} = \mathrm{e}^{\mathrm{i}2\pi k\alpha}. \tag{2}$$

Hence $\alpha = 1/(n+1)$ and

$$\frac{2\pi\nu}{n+1} = 2\pi k\alpha + 2\pi m, \tag{3}$$

where m is an integer. One simple solution is

$$\nu = k, \tag{4}$$

where we have taken $m = 0$ for simplicity, and the integer part we ignored does not influence the intermediate statistics quantum bracket and other properties. Note that $0 \leq \nu \leq n$, so the restriction on the winding number k is

$$0 \leq k \leq \frac{1}{\alpha} - 1, \tag{5}$$

where k is an integer, because α is a fraction. For example for $n = 3, \alpha = 1/4$, then $\nu = 0, 1, 2, 3$, and the values of winding number k are $0, 1, 2, 3$.

As we know, in the fractional statistics of anyon, the number of winding numbers k is infinite. When we relate the winding number representation with the occupation number representation, the number of k becomes finite. This means that the winding number is restricted by the maximum occupation number. For 1/3-anyon, the statistics parameters $\alpha = 1/3$ and the maximum occupation number $n = 2$, there are two 1/3-anyons in one quantum state, maximally. They can only wind two circles maximally. In other words, not all winding numbers are meaningful to Gentile statistics, and $0 \leq k \leq n$. The possible values of winding numbers relating to the Gentile statistics is only a subset of all possible winding numbers.

Now we address the question of the required phase factor when we braid or exchange two anyons in the same state. One can get the required phase when the two particles are distinguishable, and no phase when the two braiding particles are identical. Let's take an example from the first Kitaev mode where there are four superselection sectors: the vacuum, e, m and ε [18]. When we braid two different particles, for instance braiding e around m, we can get the required phase, $\mathrm{e}^{\mathrm{i}2\pi k\alpha}$. However for two identical particles, braiding e around e for instance, there is no phase factor. This is because in Gentile statistics, the creation operator and the annihilation operator are not $a^\dagger$ and a, but $a^\dagger$ and b. We can regard this as two different kinds of particles, and they satisfy

$$(a^\dagger b^\dagger - b^\dagger a^\dagger)|\nu\rangle_n = 2\mathrm{i}\,\mathrm{Im}\sqrt{\langle \nu+1\rangle_n^* \langle \nu+2\rangle_n}|\nu+2\rangle_n. \tag{6}$$

The equation above indicates that we can get a phase when we exchange two different particles. For two identical particles

$$(a^\dagger a^\dagger - a^\dagger a^\dagger)|\nu\rangle_n = 0, \tag{7}$$

no phase factor is created.

Notice that the transformation of representation does not change the energy spectrum, the energy spectrum of the intermediate-statistics oscillator in the winding number representation is the same as that in the occupation number representation.

3. Operators in the winding number representation

The intermediate statistics quantum bracket can be written as [21],

$$[u, v]_n \equiv uv - \mathrm{e}^{\mathrm{i}\theta_n} vu, \tag{8}$$

where n represents the maximum occupation number in one quantum state, $\theta_n = 2\pi/(n+1)$, u and v are arbitrary operators. When $n = \infty$, it becomes commutativity and when $n = 1$ it becomes anticommutativity, and Gentile statistics returns to Bose–Einstein statistics and Fermi–Dirac statistics respectively. $a^\dagger$ is the creation operator, and b is the annihilation operator, and they are not Hermitian conjugate reciprocally. They obey the relationship of the intermediate statistics quantum bracket

$$[b, a^\dagger]_n \equiv ba^\dagger - \mathrm{e}^{\frac{\mathrm{i}2\pi}{n+1}} a^\dagger b = 1. \tag{9}$$

According to Eq. (2), we can rewrite it as

$$[b, a^\dagger]_\alpha \equiv ba^\dagger - \mathrm{e}^{\mathrm{i}2\pi j\alpha} a^\dagger b = 1, \tag{10}$$

where $\alpha = 1/(n+1)$ and $j = 1$. The basic vector $|\nu\rangle_n$ increases by 1 every time when the creation operator is operated on. In the winding number representation, the case of basic vector $|k\rangle_\alpha$ is the same, it increases the winding number by one each time. The operation of the creation and annihilation operator can be written as

$$a^\dagger |k\rangle_\alpha = \sqrt{\langle k+1\rangle_\alpha}|k+1\rangle_\alpha, \tag{11}$$

$$b|k\rangle_\alpha = \sqrt{\langle k\rangle_\alpha}|k-1\rangle_\alpha, \tag{12}$$

where

$$\langle k\rangle_\alpha = \frac{1-\mathrm{e}^{\mathrm{i}2\pi\alpha k}}{1-\mathrm{e}^{\mathrm{i}2\pi\alpha}}. \tag{13}$$

We define the operator

$$B_{k\alpha} \equiv ba^\dagger - a^\dagger b, \tag{14}$$

and the effect of $B_{k\alpha}$ on a state in the winding number representation is to add a phase factor,

$$B_{k\alpha}|k\rangle_\alpha = \mathrm{e}^{\mathrm{i}2\pi k\alpha}|k\rangle_\alpha. \tag{15}$$

For this reason, we call it the winding operator. The particle number operator can be rewritten as

$$N = \frac{n+1}{2\pi} \arccos\left[\frac{1}{2}(B_{k\alpha} + B_{k\alpha}^\dagger)\right]. \tag{16}$$

In the winding number representation, according to Eq. (2), we can substitute $[u, v]_\alpha$ for $[u, v]_n$ [20] where u and v are arbitrary operators. Further, this property is valid for all the intermediate statistics quantum bracket in Ref. [20]. For example, if u, v, w are arbitrary operators, then we have the Jacobi-like identities,

$$\begin{aligned}&[[u, v]_\alpha, w]_\alpha + [[w, u]_\alpha, v]_\alpha + [[v, w]_\alpha, u]_\alpha + [[v, u]_\alpha, w]_\alpha + [[w, v]_\alpha, u]_\alpha + [[u, w]_\alpha, v]_\alpha \\ &\quad = (1-\mathrm{e}^{\mathrm{i}2\pi k\alpha})^2(uvw + wuv + vwu + vuw + wvu + uwv),\end{aligned} \tag{17}$$

and

$$\begin{aligned}&[[u, v]_\alpha, w]_\alpha + [[w, u]_\alpha, v]_\alpha + [[v, w]_\alpha, u]_\alpha - [[v, u]_\alpha, w]_\alpha - [[w, v]_\alpha, u]_\alpha - [[u, w]_\alpha, v]_\alpha \\ &\quad = (1-\mathrm{e}^{\mathrm{i}4\pi k\alpha})(uvw + wuv + vwu - vuw - wvu - uwv).\end{aligned} \tag{18}$$

4. Anyon coherent state

The coherent state is a very useful concept in physics. For anyon, it is very difficult to get a coherent state. For instance, in the first Kitaev model, Abelian anyons are created in pairs by operating the string operators on ground states or annihilated by using string operators on excited states. The string operator depends on the string and the excited state is determined by the homotopy class of string [19]. Here we give a simple expression for the anyon coherent state in the winding number representation.

By definition, the anyon coherent state is the eigenstate of annihilation operator b in the winding number representation,

$$b|\varphi\rangle_\alpha = \varphi|\varphi\rangle_\alpha. \tag{19}$$

Similar to the Fermion coherent state [28], we express the coherent state $|\varphi\rangle_\alpha$ as follows,

$$|\varphi\rangle_\alpha = M\left(|0\rangle_\alpha + \eta(1,\alpha)|k_1\rangle_\alpha\varphi + \eta(2,\alpha)|k_2\rangle_\alpha\varphi^2 + \cdots + \eta\left(\frac{1}{\alpha}-1,\alpha\right)|k_{\frac{1}{\alpha}-1}\rangle_\alpha\varphi^{\frac{1}{\alpha}-1}\right), \tag{20}$$

where M is the normalization constant, $\eta(i,\alpha)$ $(i = 1, 2\cdots 1/\alpha - 1)$ are coefficients to be determined, and $k_i = \nu_i = i$. φ is analogous to the Grassmann number satisfying

$$\varphi^{\frac{1}{\alpha}} = 0. \tag{21}$$

For example when $\alpha = 1/4$ and $n = 3$ then $\varphi^4 = 0$. φ does not commute with vector $|k_i\rangle_\alpha$. Assume their relationship as

$$\varphi|k_i\rangle_\alpha = \mathrm{e}^{\pm \mathrm{i}2\pi\alpha k_i}|k_i\rangle_\alpha\varphi. \tag{22}$$

Using Eqs. (12) and (22), the coefficients of Eq. (20) have the forms

$$\eta(j,\alpha) = \prod_{p=1}^{j}\frac{\mathrm{e}^{\pm \mathrm{i}2\pi\alpha k_{p-1}}}{\sqrt{\langle k_p\rangle_\alpha}}. \tag{23}$$

The adjoint state vector is

$$\langle\varphi|_\alpha = M(\langle 0|_\alpha + \eta(1,\alpha)^*\overline{\varphi}\langle k_1|_\alpha + \eta(2,\alpha)^*\overline{\varphi}^2\langle k_2|_\alpha + \cdots + \eta\left(\frac{1}{\alpha}-1,\alpha\right)^*\overline{\varphi}^{\frac{1}{\alpha}-1}\langle k_{\frac{1}{\alpha}-1}|_\alpha). \tag{24}$$

Notice that $\overline{\varphi}$ is not the adjoint of φ [28], they are independent. According to $\langle\varphi|\varphi\rangle = 1$, the normalization constant

$$M = \left(1 + \sum_{q=1}^{\frac{1}{\alpha}-1}\overline{\varphi}^q\varphi^q|\eta(q,\alpha)|^2\right)^{-\frac{1}{2}}. \tag{25}$$

Furthermore, according to

$$|k\rangle_\alpha = \frac{(a^\dagger)^k}{\sqrt{\langle k\rangle_\alpha!}}|0\rangle_\alpha = \frac{(b^\dagger)^k}{\sqrt{\langle k\rangle_\alpha^*!}}|0\rangle_\alpha, \tag{26}$$

where $\langle k\rangle_\alpha! = \langle k\rangle_\alpha\langle k-1\rangle_\alpha\cdots\langle 1\rangle_\alpha$. We have some properties of Grassmann number and operators:

$$\begin{aligned}
\varphi(a^\dagger)^k &= \mathrm{e}^{\pm \mathrm{i}2\pi\alpha k}(a^\dagger)^k\varphi,\\
\varphi(b^\dagger)^k &= \mathrm{e}^{\pm \mathrm{i}2\pi\alpha k}(b^\dagger)^k\varphi,\\
\varphi a^k &= \mathrm{e}^{\pm \mathrm{i}2\pi\alpha k}a^k\varphi,\\
\varphi b^k &= \mathrm{e}^{\pm \mathrm{i}2\pi\alpha k}b^k\varphi.
\end{aligned} \tag{27}$$

5. Anyons and Gentile particles with Berry Phase

In this section, we show that some Gentile particles with a Berry phase have the properties of anyons. Let us consider a Gentile particle which is undergoing a unitary transformation $U(\phi)$, where ϕ is a parameter which is varied cyclically, and the Hamiltonian of the system is $H(\phi) = U(\phi)HU^\dagger(\phi)$. We also assume that the transformation is adiabatic. After the transformation, the eigenstate acquires a Berry phase $\mathrm{e}^{\mathrm{i}\gamma_\nu}$. As an illustrative example, we adopt $U(\phi)$ by $J_z = N - n/2$ as in Ref. [21]

$$U(\phi) = \mathrm{e}^{-\mathrm{i}\phi J_z}. \tag{28}$$

In the occupation number representation of Gentile statistics, the Berry phase is

$$\begin{aligned}\gamma_\nu &= \mathrm{i}\int_c d\phi \langle \nu | U^\dagger(\phi) \nabla_\phi U(\phi) | \nu \rangle_n \\ &= 2\pi \langle \nu | J_z | \nu \rangle_n \\ &= 2\pi \left(\nu - \frac{n}{2} \right). \end{aligned} \tag{29}$$

If we let this Berry phase above equal that of the anyons, that is, $\exp(\mathrm{i}2\pi k\alpha) = \exp(\mathrm{i}\gamma_\nu)$, and according to Eq. (1) we have

$$k = (\nu - n/2)(n + 1), \tag{30}$$

where $\alpha = 1/(n + 1)$ has been adopted.

When the Berry phase of the Gentile particle equals the braiding phase of anyon, we have a restriction on k. For example, when $n = 2$, $\nu = 0, 1, 2$, then $k = 3(\nu - 1) = -3, 0, 3$; when $n = 3$, $\nu = 0, 1, 2, 3$, then $k = 4\nu - 6 = -6, -2, 0, 2, 6$. The positive values of k means clockwise braiding, and the negative values means anticlockwise braiding. When an anyon braids around another one, this phase can be interpreted as the Berry phase generated by the particles of Gentile statistics in the parameter space. Here, the transformation equation (28) describes a kind of closed path in parameter space. Of course, the closed path can be other forms, Eq. (28) is a simple one. This is another method to relate Gentile particles and anyons. In the above sections, the braiding phase of anyon equals the phase factor of the intermediate statistics quantum bracket. In this section, it equals the Berry phase of the Gentile particle. From this perspective, anyons are simulated by the particles of Gentile statistics in terms of the Berry phase.

6. Conclusions

Anyons and the particles of Gentile statistics are both subjects of intermediate-statistics. We introduced the winding number representation of anyons and proposed the intermediate statistics quantum brackets description of anyons. We related the phase factor of anyons to the maximum occupation number of Gentile particles. Then we adapted the intermediate statistics quantum bracket representation to anyons in the winding number representation, and introduced annihilation and creation operators for anyons and defined a so-called winding operator, which adds a phase factor to a state in the winding number representation. Finally, we used this new description of anyons in two ways, we give the anyon coherent state and demonstrated that anyons can be viewed as Gentile particle states with a Berry phase.

The intermediate statistics of anyons and Gentile particles have their characteristics respectively, and they are developed separately. In this paper, we gave a possible link between these two intermediate statistics. However, we should point out that the anyon statistics is not a complete analogue of the Gentile statistics. We have related their phases with a simple substitution of parameters. The results offer the possibility to characterize and understand better each of the statistics and they make available a new practical framework to describe anyons. For example, the study of some systems which obey Gentile statistics in the winding number representation is an interesting topic. It is also interesting to see if it can be applied to quantum information processing based on anyons.

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